The partition bundle of type A_{N-1} $(2,0)$ theory

Måns Henningson

Department of Fundamental Physics
Chalmers University of Technology
S-412 96 Göteborg, Sweden

mans@chalmers.se

Abstract:

Six-dimensional $(2,0)$ theory can be defined on a large class of six-manifolds endowed with some additional topological and geometric data (i.e. an orientation, a spin structure, a conformal structure, and an R-symmetry bundle with connection). We discuss the nature of the object that generalizes the partition function of a more conventional quantum theory. This object takes its values in a certain complex vector space, which fits together into the total space of a complex vector bundle (the 'partition bundle') as the data on the six-manifold is varied in its infinite-dimensional parameter space. In this context, an important role is played by the middle-dimensional intermediate Jacobian of the six-manifold endowed with some additional data (i.e. a symplectic structure, a quadratic form, and a complex structure). We define a certain hermitian vector bundle over this finite-dimensional parameter space. The partition bundle is then given by the pullback of the latter bundle by the map from the parameter space related to the six-manifold to the parameter space related to the intermediate Jacobian.

1 Introduction

Six-dimensional $(2, 0)$ theory is a comparatively new kind of quantum theory, which in many respects is rather different from how we think of e.g. quantum field theory [1]. These theories are remarkably unique, and can be completely specified by the type

$$\begin{aligned} \Phi &\in \text{ADE} \\ &\simeq \{\text{simply laced Lie algebras}\} \\ &\simeq \{\text{finite subgroups of SU}(2)\}. \end{aligned} \tag{1.1}$$

A $(2, 0)$ theory can be defined on an arbitrary six-manifold M which is orientable and admits a spin structure, i.e. the first two Stiefel-Whitney classes of its tangent bundle must vanish:

$$w_1(TM) = w_2(TM) = 0. \tag{1.2}$$

Furthermore, M must be endowed with some additional topological and geometrical data:

$$\begin{aligned} \sigma &\in \Sigma \\ &= \{\text{orientations on } M\} \\ &= \text{affine space over } H^0(M, \mathbb{Z}_2) \\ s &\in \mathcal{S} \\ &= \{\text{spin structures on } M\} \\ &= \text{affine space over } H^1(M, \mathbb{Z}_2) \\ [g] &\in \mathcal{G} \\ &= \{\text{conformal structures on } M\} \\ &= \text{infinite dimensional real manifold} \end{aligned} \tag{1.3}$$

as well as data related to the R-symmetry:

$$\begin{aligned} R &\in \mathcal{R} \\ &= \{\text{principal Sp}(4) \simeq \text{Spin}(5) \text{ bundles over M}\} \\ &= \text{discrete set labeled by characteristic classes in } H^4(M, \mathbb{Q}) \text{ and } H^5(M, \mathbb{Z}_2) \\ A &\in \mathcal{A} \\ &= \{\text{connections on } R\} \\ &= \text{affine space over } \Omega^1(M, \text{ad}(R)). \end{aligned} \tag{1.4}$$

Our main concern in this paper is to investigate, for a fixed type Φ and a fixed topological class of M, the dependence of the theory on the data $(\sigma, s, [g])$ and (R, A). In particular, we will try to elucidate the nature of an object Z that generalizes the partition function of a more conventional quantum theory.

A first remark is that, because of various 'anomalies', there is a phase ambiguity in Z:

- The conformal anomaly [2] implies a dependence on the choice of representative metric g for the conformal structure $[g] \in \mathcal{G}$.

- The chiral, gravitational and mixed chiral/gravitational anomalies [3, 4] imply a dependence on the choice of trivialization of the Sp(4) bundle R and the parametrization of M.

These phenomena clearly provide extremely important clues for a future complete definition of $(2,0)$ theory, but will nevertheless be disregarded in this paper. Our ambition here is thus only to describe the nature of Z up to a complex 'anomalous' phase factor. (Even so, we are not attempting to actually determine Z, although we hope that this will eventually be possible; in this paper we will just answer the question of what kind of object Z is.)

For simplicity, we will restrict ourselves to the case where R is a trivial bundle and A is the trivial connection. Furthermore, we will only consider $(2,0)$ theory of type $\Phi = A_{N-1}$ for $N = 2, 3, \ldots$, since this allows us to use results from holography [5]. But we hope to continue to more general cases in the near future.

The next important point is that, for fixed data $(\sigma, s, [g])$, Z is not a single complex number, but rather an element of a certain finite-dimensional complex vector space V [6]. (This should not be confused with the more familiar quantum mechanical Hilbert space $\mathcal{H}$, that appears when the theory is considered on a manifold M with a prefered (Euclidean) 'time' direction). Indeed, the data $(\sigma, s, [g])$ on M determine data related to the intermediate Jacobian

$$T = H^3(M, \mathbb{R})/H^3(M, \mathbb{Z}), \tag{1.5}$$

which is a torus of dimension

$$\dim_{\mathbb{R}} T = 2n = b_3(M). \tag{1.6}$$

These data are:

$$\begin{aligned}
\omega &\in \Omega \\
&= \{\text{symplectic structures on } H^3(M, \mathbb{R}) \text{ induced from the intersection form}\} \\
&= \text{set with 2 elements} \\
u &\in \mathcal{U} \\
&= \{\text{non}-\text{degenerate quadratic forms on } H^3(M, \mathbb{Z}_2) \text{ polarized by } \omega\} \\
&= \text{set with } 2^{2\mathrm{n}} \text{ elements} \\
J &\in \mathcal{J} \\
&= \{\text{translation invariant complex structures on } T\} \\
&= \text{complex space of dimension } \frac{1}{2} n(n+1).
\end{aligned} \tag{1.7}$$

The data (ω, u, J) define a hermitian line-bundle $\mathcal{L}$ over T [7, 8], and for $(2,0)$ theory of type $\Phi = A_{N-1}$, the complex vector space V in which Z takes its values can be identified with the space $H^0(T, \mathcal{L}^N)$ of holomorphic sections of $\mathcal{L}^N$ [6]. These matters are reviewed in the next section.

In section three, we describe how the family of vector spaces V fit together into the total space of a complex vector bundle over the parameter space $\Sigma \times \mathcal{S} \times \mathcal{G}$ as the data $(\sigma, s, [g])$ are varied. Z should thus be understood as the 'partition section'

of this ‘partition bundle’. (This is largely implicit in earlier work on $(2,0)$ theory [6, 9]; our aim here is merely to make these results somewhat more explicit.) In fact, the family of spaces $H^0(T, \mathcal{L}^N)$ naturally fit together to the total space of a hermitian vector bundle over the finite-dimensional parameter space $\Omega \times \mathcal{U} \times \mathcal{J}$ in which the data (ω, u, J) take its values. More precisely, for fixed values of $\omega \in \Omega$ and $u \in \mathcal{U}$, we can define a hermitian vector bundle $\tilde{E}$ of rank N^n over $\overline{\mathcal{J}}/\Lambda_{(\omega,u)}$, where $\overline{\mathcal{J}}$ denotes the universal covering space of $\mathcal{J}$, and $\Lambda_{(\omega,u)}$ is the subgroup of the automorphism group of $H^3(M,\mathbb{Z})$ that leaves these data invariant. We have a map

$$\phi\colon \overline{\mathcal{G}}/\Lambda_{(\sigma,s)} \to \overline{\mathcal{J}}/\Lambda_{(\omega,u)} \tag{1.8}$$

which determines the complex structure J on T in terms of the conformal structure $[g]$ on M. Here $\overline{\mathcal{G}}$ is the universal covering space of $\mathcal{G}$, and $\Lambda_{(\sigma,s)}$ is the subgroup of the mapping class group of M that stabilizes the data $\sigma \in \Sigma$ and $s \in \mathcal{S}$. The partition bundle E is now the complex vector bundle over the infinite dimensional space $\overline{\mathcal{G}}/\Lambda_{(\sigma,s)}$ given by the pullback of $\tilde{E}$ by the map ϕ:

$$E = \phi^*(\tilde{E}). \tag{1.9}$$

In section four, we make these somewhat abstract considerations more concrete by choosing a parametrization of $\overline{\mathcal{J}}$ so that it can be identified with a Siegel generalized upper half-space. This allows us to determine the transition functions of the hermitian vector bundle $\tilde{E}$ over the moduli space $\overline{\mathcal{J}}/\Lambda_{(\omega,u)}$. We find that these are related to a Siegel modular form and a flat vector bundle.

In an appendix, we exemplify the determination of the quadratic form u for all possible choices of spin structure $s \in \mathcal{S}$ in the particular case of $M = T^6$.

2 The vector space

This section is much influenced by [7, 6].

The symplectic structure $\omega \in \Omega^2(T)$ on the middle dimensional intermediate Jacobian $T = H^3(M,\mathbb{R})/H^3(M,\mathbb{Z})$ of M only depends on the orientation $\sigma \in \Sigma$ on M and is given by

$$\omega[\delta_1 C, \delta_2 C] = \int_M \delta_1 C \wedge \delta_2 C. \tag{2.10}$$

Here $\delta_1 C, \delta_2 C \in H^3(M,\mathbb{R})$ are regarded as tangent vectors to T. Clearly, ω is closed (in fact constant) and

$$\int_T \frac{1}{n!}\omega^n = 1, \tag{2.11}$$

so the de Rham cohomology class $[\omega] \in H^2_{\mathrm{de\,Rham}}(T)$ lies in the image of the inclusion $H^2(T,\mathbb{Z}) \subset H^2(T,\mathbb{R}) \simeq H^2_{\mathrm{de\,Rham}}(T)$.

By the de Rham theorem and the Hodge theorem

$$H^3(M,\mathbb{R}) \simeq H^3_{\mathrm{de\,Rham}}(M) \simeq \Omega^3_{\mathrm{harmonic}}(M), \tag{2.12}$$

where $\Omega^3_{\text{harmonic}}(M)$ are the three-forms on M that are harmonic with respect to the conformal structure $[g]$ on M. Together with the orientation $\sigma \in \Sigma$ on M, $[g] \in \mathcal{G}$ determines the Hodge duality operator

$$* \colon \Omega^3_{\text{harmonic}}(M) \to \Omega^3_{\text{harmonic}}(M). \tag{2.13}$$

This operator obeys $** = -1$, and thus defines a complex structure $J \in \mathcal{J}$ on T, i.e. we have (for given $\sigma \in \Sigma$) a map

$$\phi \colon \mathcal{G} \to \mathcal{J}. \tag{2.14}$$

The symplectic structure $\omega \in \Omega^2(T)$ is of type $(1,1)$ with respect to the conformal structures defined in this way, and thus gives T the structure of a (flat) Kähler manifold.

We will now explain how the spin structure $s \in \mathcal{S}$ on M determines a quadratic form

$$u \colon H^3(M, \mathbb{Z}_2) \to \mathbb{Z}_2. \tag{2.15}$$

A straight line from 0 to an element $\gamma \in H^3(M, \mathbb{Z})$ descends to a closed curve on T, and thus determines a three-form gauge-field $C \in \Omega^3(S^1 \times M)$ with field-strength $G = dC$. We let X be an eight-dimensional open spin manifold, which bounds $S^1 \times M$ and over which the product of the anti-periodic spin structure on S^1 and the given spin structure on M extends. (By a result in cobordism theory, such a manifold always exists.) Extending G to X we then define $u(\gamma) \in \mathbb{Z}_2$ by

$$(-1)^{u(\gamma)} = \exp\left(2\pi i \frac{1}{2} \int_X (G \wedge G - \frac{p_1}{2} \wedge \frac{p_1}{2})\right), \tag{2.16}$$

where p_1 denotes the first Pontryagin class of TX (which can be divided by two in a canonical way for X spin). Clearly, this expression only depends on the reduction of γ modulo two. Furthermore, it is in fact well-defined and independent of the choice of X despite the half-integer prefactor [10]. Finally, it obeys the condition

$$u(\gamma) + u(\gamma') = \omega[\gamma, \gamma'] + u(\gamma + \gamma'), \tag{2.17}$$

for $\gamma, \gamma' \in H^3(M, \mathbb{Z}_2)$, i.e. the symplectic structure ω acts as a polarization of u.

The data (ω, u, J) determine a hermitian line bundle $\mathcal{L}$ over T: Its curvature equals the symplectic structure ω, and the holonomy along the closed curve defined by a straight line from 0 to $\gamma \in H^3(M, \mathbb{Z})$ is given by the quadratic form u as $(-1)^{u(\gamma)}$. We will be mostly interested in the Nth power $\mathcal{L}^N$ of $\mathcal{L}$. This bundle has curvature $N\omega$. The holonomies along the curves γ described above are all trivial for even N (in which case $\mathcal{L}^N$ is actually independent of the spin structure $s \in \mathcal{S}$), and agree with the holonomies of $\mathcal{L}$ for odd N.

For $(2,0)$ theory of type $\Phi = A_{N-1}$, the vector space V in which Z takes its values can now be determined by geometric quantization as the Hilbert space of a certain topological field theory of Schwarz type [11]. This field theory governs the three-form gauge field C that is part of the holographic gravity dual of the $(2,0)$

theory on an open seven manifold Y that bounds M. In this way, one finds that V is given by the space of holomorphic sections of the bundle $\mathcal{L}^N$:

$$V \simeq H^0(T, \mathcal{L}^N). \tag{2.18}$$

The Hilbert space inner product on V is

$$\langle s|s'\rangle = \int_T \frac{1}{n!}\omega^n\,(s, s') \tag{2.19}$$

for $s, s' \in H^0(T, \mathcal{L}^N)$, where $(.,.)$ is the Hermitian structure on the fibers of $\mathcal{L}$.

To avoid a possible misunderstanding, we point out that this does not mean that the $(2,0)$ theory on $M = \partial Y$ couples to a three-form gauge field C. Indeed, in a situation with a stack of N parallel $M5$-branes in M-theory, the C-field of M-theory couples to the collective 'center of mass' degrees of freedom of the branes rather than to the $(2,0)$ theory of type $\Phi = A_{N-1}$ defined on the world-volume of the branes. This is analogous to a situation with a stack of N parallel $D3$-branes in type IIB string theory, where the Yang-Mills theory on the branes has gauge group $\mathrm{SU}(N)$ rather than $\mathrm{U}(N) \simeq (\mathrm{SU}(N)\times\mathrm{U}(1))/\sim$. (The quotient is by the equivalence relation $(\eta\mathbb{1}_N, \eta^{-1}) \sim (\mathbb{1}_N, 1)$ for η an Nth root of unity.) Here it is the $U(1)$ factor that represents the collective degrees of freedom.

Returning to the vector space $V \simeq H^0(T, \mathcal{L}^N)$, it follows from the Kodaira vanishing theorem, that the higher cohomology groups $H^k(T, \mathcal{L}^N)$ for $1 \leq k \leq n = \dim_{\mathbb{C}} T$ vanish. The Hirzebruch-Riemann-Roch formula then gives

$$\begin{aligned}
\dim V &= \sum_{k=0}^{n}(-1)^k \dim H^k(T, \mathcal{L}^N) \\
&= \int_T e^{c_1(\mathcal{L}^N)}\mathrm{Td}(T) \\
&= \int_T \frac{1}{n!}(N\omega)^n \\
&= N^n,
\end{aligned} \tag{2.20}$$

where we have used that the total Todd character $\mathrm{Td}(T) = 1$ for the flat manifold T and first the Chern class $c_1(\mathcal{L}^N) = N[\omega]$.

The bundle $\mathcal{L}$ over $T = H^3(M,\mathbb{R})/H^3(M,\mathbb{Z})$ has no non-trivial isometries, but $\mathcal{L}^N$ is invariant under translations

$$T_c \colon T \to T \tag{2.21}$$

by elements $c \in \frac{1}{N}H^3(M,\mathbb{Z}) \subset H^3(M,\mathbb{R})$. Obviously $T_c^N = \mathrm{id}$ and $T_c T_{c'} = T_{c'} T_c$. But the induced pullback maps

$$T_c^* \colon H^0(T, \mathcal{L}^N) \to H^0(T, \mathcal{L}^N) \tag{2.22}$$

instead obey the relations

$$(T_c^*)^N = (-1)^{u(Nc)} \tag{2.23}$$

and the Heisenberg relations

$$T_c^* T_{c'}^* = T_{c'}^* T_c^* \exp\left(N c \wedge c'\right). \tag{2.24}$$

In the last formula, we have used the convenient short-hand notation

$$\exp(v) = \exp\left(2\pi i \int_M v\right) \tag{2.25}$$

for $v \in H^6(M, \mathbb{C})$.

3 The vector bundle

There is a natural homomorphism from the mapping class group of M to the subgroup, isomorphic to $\mathrm{Sp}_{2n}(\mathbb{Z})$, of the automorphism group of $H^3(M, \mathbb{Z})$ that stabilizes the symplectic structure ω. This symplectic subgroup permutes the possible quadratic forms u fulfilling (2.17) via its reduction modulo two, which is isomorphic to the group $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ of order

$$|\mathrm{Sp}_{2n}(\mathbb{Z}_2)| = 2^{n^2} \prod_{i=1}^{n} (2^{2i} - 1). \tag{3.26}$$

There are two orbits under this action:

The first orbit consist of quadratic forms u for which $H^3(M, \mathbb{Z})$ admits a decomposition

$$H^3(M, \mathbb{Z}) = A \oplus B \tag{3.27}$$

such that

$$\int_M n \wedge n' = \int_M m \wedge m' = 0 \tag{3.28}$$

for $n, n' \in A$ and $m, m' \in B$ and furthermore

$$u(a + b) = \int_M a \wedge b \tag{3.29}$$

for $a \in A \otimes \mathbb{Z}_2$ and $b \in B \otimes Z_2$. This means that u gives $H^3(M, \mathbb{Z}_2)$ the structure of the direct sum of n hyperbolic planes. We have

$$u(\gamma) = \left\{ \begin{array}{lll} 0 & \text{for } 2^{2n-1} + 2^n - 2^{n-1} & \text{values of } \gamma \in H^3(M, \mathbb{Z}_2) \\ 1 & \text{for } 2^{2n-1} - 2^{n-1} & \text{values of } \gamma \in H^3(M, \mathbb{Z}_2). \end{array} \right. \tag{3.30}$$

(The sum of these numbers of course equals the cardinality 2^{2n} of $H^3(M, \mathbb{Z}_2)$.) The stabilizer in $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$ of such a quadratic form u is isomorphic to the group $\mathrm{O}_{2n}^+(\mathbb{Z}_2)$ of order

$$|\mathrm{O}_{2n}^+(\mathbb{Z}_2)| = 2 \cdot 2^{n(n-1)} (2^n - 1) \prod_{i=1}^{n-1} (2^{2i} - 1), \tag{3.31}$$

so the cardinality of the orbit is

$$\frac{|\mathrm{Sp}_{2n}(\mathbb{Z}_2)|}{|\mathrm{O}^+_{2n}(\mathbb{Z}_2)|} = 2^{2n-1} + 2^{n-1}. \tag{3.32}$$

The second orbit consist of quadratic forms u for which (3.29) is replaced by

$$u(a+b) = \int_M a \wedge b + (\int_M a \wedge b_s)^2 + (\int_M a_s \wedge b)^2 \tag{3.33}$$

for some non-zero $a_s \in A \otimes \mathbb{Z}_2$ and $b_s \in B \otimes \mathbb{Z}_2$. $H^3(M, \mathbb{Z}_2)$ now has the structure of the direct sum of $n-1$ hyperbolic planes and a two-dimensional anisotropic space. There number of values of $\gamma \in H^3(M, \mathbb{Z}_2)$ for which $u(\gamma) = 0$ or $u(\gamma) = 1$ are interchanged as compared to the quadratic forms on the first orbit. The stabilizer of u is isomorphic to the group $\mathrm{O}^-_{2n}(\mathbb{Z}_2)$ of order

$$|\mathrm{O}^-_{2n}(\mathbb{Z}_2)| = 2 \cdot 2^{n(n-1)}(2^n + 1) \prod_{i=1}^{n-1} (2^{2i} - 1), \tag{3.34}$$

so the cardinality of the orbit is

$$\frac{|\mathrm{Sp}_{2n}(\mathbb{Z}_2)|}{|\mathrm{O}^-_{2n}(\mathbb{Z}_2)|} = 2^{2n-1} - 2^{n-1}. \tag{3.35}$$

The sum of the cardinalities of the two orbits of course equal the cardinality 2^{2n} of the set of all quadratic forms obeying (2.17). But for a given manifold M, not all such quadratic forms need to appear for any spin structure $s \in \mathcal{S}$ on M. (Obviously, at most $2^{b_1(M)}$ different quadratic forms can appear, which gives a restriction for a manifold M such that $b_1(M) \leq b_3(M)$.) I am not aware of any example from the second orbit, and I conjecture that in fact only the first orbit can appear. A proof of this (or a counterexample) would be interesting. In any case, for the remainder of this paper, we will only consider the first orbit, which certainly leads to the nicest looking formulas.

In the next section, we will construct a basis for the vector space $V \simeq H^0(T, \mathcal{L}^N)$ that depends on a lifting of $J \in \mathcal{J}$ to the universal covering space $\overline{\mathcal{J}}$. This defines a global holomorphic frame for a holomorphic vector bundle over the topologically trivial space $\overline{\mathcal{J}}$. (This bundle is of course trivial.) The $\mathrm{Sp}_{2n}(\mathbb{Z})$ subgroup of the automorphism group of $H^3(M, \mathbb{Z})$ that stabilizes ω acts on $\overline{\mathcal{J}}$. It follows from the above that the subgroup $\Lambda_{(\omega,u)}$ that also stabilizes a given quadratic form $u \in \mathcal{U}$ (on the first orbit above) is isomorphic to the kernel of the composite homomorphism

$$p \colon \mathrm{Sp}_{2n}(\mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z}_2) \to \mathrm{Sp}_{2n}(\mathbb{Z}_2)/\mathrm{O}^+_{2n}(\mathbb{Z}_2), \tag{3.36}$$

where the first map is reduction modulo two, and the second map is the quotient projection. Thus

$$\Lambda_{(\omega,u)} \simeq \ker p. \tag{3.37}$$

In this way, we get a topologically non-trivial holomorphic vector bundle $\tilde{E}$ over the quotient space $\overline{\mathcal{J}}/\Lambda_{(\omega,u)}$. As described in the introduction, pullback by $\phi \colon \overline{\mathcal{G}}/\Lambda_{(\sigma,s)} \to \overline{\mathcal{J}}/\Lambda_{(\omega,u)}$ defines the partition bundle E over $\overline{\mathcal{G}}/\Lambda_{(\sigma,s)}$ of which Z is a section.

4 The trivialization

The universal cover $\overline{\mathcal{J}}$ of the space of flat Kähler metrics on T can be identified with the genus n Siegel upper half space, i.e. the space of complex, symmetric $n \times n$ matrices with positive definite imaginary part. Much of what follows is analogous to the theory of holomorphic modular forms for (a subgroup of) $\mathrm{Sp}_{2n}(\mathbb{Z})$ (see e.g. [17]), but since we prefer to work in a differential form notation instead of choosing a specific basis for $H^3(M, \mathbb{R})$, some formulas might look slightly unfamiliar.

We begin by defining the conjugate S^* of a linear map S from (a subspace of) $H^3(M, \mathbb{C})$ to (a subspace of) $H^3(M, \mathbb{C})$ by the requirement that

$$\int_M x \wedge Sx' = \int_M S^* x \wedge x' \tag{4.38}$$

for all applicable $x, x' \in H^3(M, \mathbb{C})$. The complex structure on T can then be described by a map

$$\tau \colon A \to B \otimes \mathbb{C} \tag{4.39}$$

which is anti self-conjugate, i.e.

$$\tau = -\tau^*, \tag{4.40}$$

and has positive definite imaginary part, i.e.

$$\frac{1}{2i} \int_M n \wedge (\tau - \bar{\tau}) n \geq 0 \tag{4.41}$$

for all $n \in A$. The intermediate Jacobian $T = H^3(M, \mathbb{R})/H^3(M, \mathbb{Z})$ can then be identified as

$$T \simeq \frac{B \otimes \mathbb{C}}{B \oplus \tau A}. \tag{4.42}$$

The space $V \simeq H^0(T, \mathcal{L}^N)$ can be identified with the space of holomorphic functions

$$\psi(\tau| \, . \,) \colon B \otimes \mathbb{C} \to \mathbb{C} \tag{4.43}$$

which satisfy the double quasi-periodicity conditions

$$\psi(\tau|z + m + \tau n) = \psi(\tau|z) \exp\left(-\frac{N}{2} n \wedge \tau n - N n \wedge z \right) \tag{4.44}$$

for $z \in B \otimes \mathbb{C}$, $n \in A$, and $m \in B$. In this trivialization of $\mathcal{L}^N$, the transition functions thus depend holomorphically on $z \in B \otimes \mathbb{C}$. Alternatively, we can work in a unitary trivialization of $\mathcal{L}^N$, with the sections given by functions $\Psi(\tau, \bar{\tau}| \, . \, , \, . \,)$ whose transition functions are $U(1)$-valued:

$$\begin{aligned} &\Psi(\tau, \bar{\tau}|z + m + \tau n, \bar{z} + m + \bar{\tau} n) = \Psi(\tau, \bar{\tau}|z, \bar{z}) \\ &\times \exp\left(\frac{N}{2} n \wedge m + \frac{N}{2} (\tau - \bar{\tau})^{-1} (z - \bar{z}) \wedge m + \frac{N}{2} \left(\tau (\tau - \bar{\tau})^{-1} \bar{z} - \bar{\tau} (\tau - \bar{\tau})^{-1} z \right) \wedge n \right). \end{aligned} \tag{4.45}$$

This makes it easy to verify the holonomies of $\mathcal{L}^N$ along the curves defined by straight lines from 0 to a point $\gamma = n+m \in H^3(M,\mathbb{Z}) = A \oplus B$: Putting $z = \bar{z} = 0$, we get

$$\Psi(\tau,\bar{\tau}|m+\tau n, m+\bar{\tau}n) = \Psi(\tau,\bar{\tau}|0,0)(-1)^{u(n+m)} \tag{4.46}$$

with

$$(-1)^{u(n+m)} = \exp\left(\frac{N}{2} n \wedge m\right). \tag{4.47}$$

In particular, the holonomies are indeed trivial for $\gamma \in A$ or $\gamma \in B$. The relationship between the two trivializations $\psi(\tau|\,.\,)$ and $\Psi(\tau,\bar{\tau}|\,.\,,.\,)$ is

$$\Psi(\tau,\bar{\tau}|z,\bar{z}) = \psi(\tau|z)\exp\left(\frac{N}{2}(\tau-\bar{\tau})^{-1}(z-\bar{z})\wedge z\right), \tag{4.48}$$

but henceforth, we will only use the holomorphic trivialization $\psi(\tau|\,.\,)$.

The hermitian inner product on $H^0(T,\mathcal{L})$ is given by

$$\begin{aligned} \langle\psi|\psi'\rangle &= \int_T d^n z\, d^n \bar{z}\, \overline{\Psi(\tau,\bar{\tau}|z,\bar{z})}\Psi'(\tau,\bar{\tau}|z,\bar{z}) \\ &= \int_T d^n z\, d^n \bar{z}\, \overline{\psi(\tau|z)}\psi'(\tau|z)\exp\left(\frac{N}{2}(\tau-\bar{\tau})^{-1}(z-\bar{z})\wedge(z-\bar{z})\right) \end{aligned} \tag{4.49}$$

We will now construct a holomorphic frame $\{\psi_{[a]}\}_{[a]\in\frac{1}{N}A/A}$ for $H^0(T,\mathcal{L}^N)$, characterized by its behavior under translations:

$$\psi_{[a]}(\tau|z+b'+\tau a') = \psi_{[a+a']}(\tau|z)\exp\left(-\frac{N}{2}a'\wedge\tau a' - Na'\wedge z + Na\wedge b'\right) \tag{4.50}$$

for $a' \in \frac{1}{N}A$ and $b' \in \frac{1}{N}B$. In particular, $\psi_{[a]}$ is an eigensection of $T_{b'}$ with eigenvalue $\exp(Na\wedge b')$. Note that (4.44) follows from (4.50) by taking $a' = n \in A$ and $b' = m \in B$. These properties determine the $\psi_{[a]}$ uniquely, up to a common holomorphic τ-dependent factor. A convenient choice is

$$\psi_{[a]}(\tau|z) = \frac{1}{\theta(\tau|0)}\sum_{n\in A}\exp\left(\frac{N}{2}(n+a)\wedge\tau(n+a) + N(n+a)\wedge z\right). \tag{4.51}$$

Here $\theta(\tau|0)$ is the Riemann theta function evaluated at $z=0$:

$$\theta(\tau|z) = \sum_{n\in A}\exp\left(n\wedge\tau n + n\wedge z\right). \tag{4.52}$$

(This is of course the unique holomorphic section of the bundle $\mathcal{L}$.) [1]

With a decomposition $H^3(M,\mathbb{Z}) = A\oplus B$ obeying (3.28) and (3.29), a general map $S\colon H^3(M,\mathbb{Z}) \to H^3(M,\mathbb{Z})$ and its conjugate map $S^*\colon H^3(M,\mathbb{Z}) \to H^3(M,\mathbb{Z})$ can be written as a matrices of maps

$$S = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} : \begin{pmatrix} B\to B & A\to B\\ B\to A & A\to A\end{pmatrix} \tag{4.53}$$

[1]There is a large literature on the connection between $(2,0)$ theory and theta functions, see e.g. [7, 12, 13, 14, 15].

and

$$S^* = \begin{pmatrix} \delta^* & \beta^* \\ \gamma^* & \alpha^* \end{pmatrix} : \begin{pmatrix} B \to B & A \to B \\ B \to A & A \to A \end{pmatrix}. \tag{4.54}$$

We say that S is symplectic if it preserves the symplectic structure ω, i.e. if

$$S^* S = S S^* = 1. \tag{4.55}$$

This amounts to the relations

$$\begin{aligned} \delta^*\alpha + \beta^*\gamma &= 1 \\ \delta^*\beta + \beta^*\delta &= 0 \\ \alpha^*\gamma + \gamma^*\alpha &= 0 \end{aligned} \tag{4.56}$$

and

$$\begin{aligned} \alpha\delta^* + \beta\gamma^* &= 1 \\ \beta\alpha^* + \alpha\beta^* &= 0 \\ \gamma\delta^* + \delta\gamma^* &= 0. \end{aligned} \tag{4.57}$$

(The unfamiliar signs are due to the definition of conjugate maps with respect to the symplectic structure.) The group of such symplectic maps is isomorphic to $\mathrm{Sp}_{2n}(\mathbb{Z})$.

A symplectic map S acts on $\tau\colon A \to B \otimes \mathbb{C}$ and $z \in B \otimes \mathbb{C}$ according to

$$\begin{aligned} \tau &\mapsto S\tau = (\alpha\tau + \beta)(\gamma\tau + \delta)^{-1} \\ z &\mapsto Sz = (\gamma\tau + \delta)^{*-1} z. \end{aligned} \tag{4.58}$$

It is useful to note that

$$S(b + \tau a) = \alpha b - \beta a + (S\tau)(-\gamma b + \delta a) \tag{4.59}$$

for $b \in B \otimes \mathbb{C}$ and $a \in A \otimes \mathbb{C}$. If ψ is a τ-dependent family of holomorphic sections obeying (4.44) for all τ, we can then define another such family $S\psi$ by

$$S\psi(\tau|z) = \psi\left(S\tau|Sz\right) \exp\left(-\frac{N}{2}\gamma z \wedge Sz\right). \tag{4.60}$$

To verify that $S\psi$ also obeys (4.44), we need to use the relations

$$\begin{aligned} \exp\left(\frac{N}{2}\delta n \wedge \beta n\right) &= 1 \\ \exp\left(\frac{N}{2}\gamma m \wedge \alpha m\right) &= 1 \\ \exp\left(N\gamma m \wedge \beta n\right) &= 1. \end{aligned} \tag{4.61}$$

The last of these three conditions is clearly always satisfied, and the first two impose non-trivial restrictions on S only for odd N. In fact, they then define the condition for the reduction of S modulo two to lie in the subgroup $O^+_{2n}(\mathbb{Z}_2)$ of $\mathrm{Sp}_{2n}(\mathbb{Z}_2)$.

We now apply this construction to the functions $\psi_{[a]}$. The new functions $S\psi_{[a]}$ obey the translation properties

$$\begin{aligned} S\psi_{[a]}(\tau|z + b' + \tau a') &= S\psi_{[a-\gamma b'+\delta a']}(\tau|z) \\ &\times \exp\left(-\frac{N}{2}a' \wedge \tau a' - Na' \wedge z + Na \wedge (\alpha b' - \beta a') \right. \\ &\left. -\frac{N}{2}\delta a' \wedge \beta a' - N\beta a' \wedge \gamma b' - \frac{N}{2}\gamma b' \wedge \alpha b'\right). \end{aligned} \tag{4.62}$$

Since both $\{\psi_{[a]}\}_{[a]\in\frac{1}{N}A/A}$ and $\{S\psi_{[a]}\}_{[a]\in\frac{1}{N}A/A}$ are frames for $H^0(T, \mathcal{L}^N)$, there must be an invertible linear relationship between them. Indeed [16], to obey (4.50), $\psi_{[a]}(\tau|z)$ must be given by a (possibly τ-dependent) multiple of

$$\begin{aligned} &\frac{1}{N^n} \sum_{[b]\in\frac{1}{N}B/B} S\psi_{[0]}(\tau|z + b + \tau a) \exp\left(\frac{N}{2}a \wedge \tau a + Na \wedge z\right) \\ &= \frac{1}{N^n} \sum_{[b]\in\frac{1}{N}B/B} S\psi_{[-\gamma b+\delta a]}(\tau|z) \times \exp\left(-\frac{N}{2}\delta a \wedge \beta a - N\beta a \wedge \gamma b - \frac{N}{2}\gamma b \wedge \alpha b\right) \end{aligned} \tag{4.63}$$

But with the choice of prefactor in (4.51), this multiple is in fact a constant given by an eight root of unity. (It depends on the transformation S, but not on $[a]$ or $[b]$, and is not given by any elementary expression. An analogous prefactor appears already in the transformation law of the Riemann theta function [17].) This can be verified by checking the special cases when

$$S = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \tag{4.64}$$

or

$$S = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \tag{4.65}$$

which generate the whole group of transformations. The partition bundle is thus related both to a holomorphic modular form of weight 1/2 (the Riemann theta function) and a flat vector bundle (the constant transition functions in (4.63)).

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A Spin structures and quadratic forms for $M = T^6$

We have

$$b_1(T^6) = 6 \tag{A.66}$$

and

$$b_3(T^6) = 2n = 20. \tag{A.67}$$

The set $\mathcal{S}$ of spin structures on T^6 is in a natural way isomorphic to the linear space $H^1(T^6, \mathbb{Z}_2)$ over $\mathbb{Z}_2$ rather than being an affine space over it. Indeed, the

SO(6) holonomies of a (flat) six torus are trivial, and a lifting of them to Spin(6) is described by stating whether they are given by a trivial or non-trivial element for the different cycles. The trivial spin structure $s_0 \in \mathcal{S}$, for which all these liftings are trivial, then corresponds to the zero element $0 \in H^1(T^6, \mathbb{Z}_2)$.

The $\mathrm{SL}_6(\mathbb{Z})$ mapping class group of T^6 acts by permutation on the set $\mathcal{S} \simeq H^1(T^6, \mathbb{Z}_2)$ of spin structures via its reduction modulo two $\mathrm{SL}_6(\mathbb{Z}_2)$. The orbits are:

rank	representative	stabilizer	cardinality
0	0	$\mathrm{SL}_6(\mathbb{Z}_2)$	1
1	e^1	$\mathrm{SL}_5(\mathbb{Z}_2) \ltimes \mathbb{Z}_2^5$	$\underline{63}$
			$2^6 = 64$

(A.68)

Here $e^1, \ldots, e^6$ is a basis of $H^1(T^6, \mathbb{Z})$, and the order of the group $\mathrm{SL}_d(\mathbb{Z}_2)$ is

$$|\mathrm{SL}_d(\mathbb{Z}_2)| = 2^{d(d-1)/2} \prod_{i=2}^{d} (2^i - 1). \tag{A.69}$$

For a given spin structure $s \in \mathcal{S}$, we now wish to determine the corresponding quadratic form

$$u \colon H^3(T^6, \mathbb{Z}_2) \to \mathbb{Z}_2. \tag{A.70}$$

Although it is not really necessary for our purposes, we start by listing the orbits of $\mathrm{SL}_6(\mathbb{Z}_2)$ on $H^2(T^6, \mathbb{Z}_2)$:

rank	representative	stabilizer	cardinality
0	0	$\mathrm{SL}_6(\mathbb{Z}_2)$	1
1	e^1e^2	$\mathrm{SL}_4(\mathbb{Z}_2) \times \mathrm{SL}_2(\mathbb{Z}_2) \ltimes \mathbb{Z}_2^8$	651
2	$e^1e^2 + e^3e^4$	$\mathrm{Sp}_4(\mathbb{Z}_2) \times \mathrm{SL}_2(\mathbb{Z}_2) \times \mathbb{Z}_2^8$	18228
3	$e^1e^2 + e^3e^4 + e^5e^6$	$\mathrm{Sp}_6(\mathbb{Z}_2)$	$\underline{13888}$
			$2^{15} = 32768$

(A.71)

More to the point is to determine the orbits of $\mathrm{SL}_6(\mathbb{Z}_2)$ on $H^3(T^6, \mathbb{Z}_2)$:

rank	representative	stabilizer	cardinality
0	0	$\mathrm{SL}_6(\mathbb{Z}_2)$	1
1	$e^1e^2e^3$	$\mathrm{SL}_3(\mathbb{Z}_2) \times \mathrm{SL}_3(\mathbb{Z}_2) \ltimes \mathbb{Z}_2^9$	1395
2	$(e^1e^2 + e^3e^4)e^5$	$\mathrm{Sp}_4(\mathbb{Z}_2) \ltimes \mathbb{Z}_2^9$	54684
2	$e^1e^2e^3 + e^4e^5e^6$	$\mathrm{SL}_3(\mathbb{Z}_2) \times \mathrm{SL}_3(\mathbb{Z}_2) \ltimes \mathbb{Z}_2$	357120
3	$e^1e^2e^6 + e^2e^3e^4 + e^1e^3e^5$	$\mathrm{SL}_3(\mathbb{Z}_2) \ltimes \mathbb{Z}_2^8$	468720
4	$e^1e^2e^6 + e^2e^3e^4 + e^1e^3e^5 + e^4e^5e^6$	order $= 2^7 \cdot 3^3 \cdot 5 \cdot 7$	$\underline{166656}$
			$2^{20} = 1048576$

(A.72)

Beginning with the $\mathrm{SL}_6(\mathbb{Z}_2)$ invariant trivial spin structure $s_0 = 0 \in H^1(T^6, \mathbb{Z}_2)$, the values of the corresponding quadratic form u_0 when evaluated on $\gamma \in H^3(T^6, \mathbb{Z}_2)$ can only depend on the orbit of γ under $\mathrm{SL}_6(\mathbb{Z}_2)$. According to (3.30), we should have

$$u_0(\gamma) = \begin{cases} 0 & \text{for } 524800 \quad \text{values of } \gamma \in H^3(T^6, \mathbb{Z}_2) \\ 1 & \text{for } 523776 \quad \text{values of } \gamma \in H^3(M, \mathbb{Z}_2). \end{cases} \tag{A.73}$$

The only solution is

$$u_0(\gamma) = \begin{cases} 1 & \text{for } \gamma \text{ in orbit of } e^1e^2e^3 + e^4e^5e^6 \\ 1 & \text{for } \gamma \text{ in orbit of } e^1e^2e^6 + e^2e^3e^4 + e^1e^3e^5 + e^4e^5e^6 \\ 0 & \text{otherwise.} \end{cases} \tag{A.74}$$

One can check that this is consistent with the condition (2.17). Continuing with a general spin structure $s \in \mathcal{S}$, the difference $\delta = u - u_0$ between the corresponding quadratic form u and the quadratic form u_0 pertaining to the trivial spin structure s_0 defines a *homomorphism*

$$\delta \colon H^3(T^6, \mathbb{Z}_2) \to \mathbb{Z}_2. \tag{A.75}$$

By Poincarè duality, such a δ can be identified with an element of $H^3(T^6, \mathbb{Z}_2)$. Clearly, $s = s_0$, i.e. $\delta = 0$, then corresponds to the element $0 \in H^3(T^6, \mathbb{Z}_2)$. But since there is no $\mathrm{SL}_6(\mathbb{Z}_2)$ orbit of cardinality $64 - 1 = 63$ on $H^3(T^6, \mathbb{Z}_2)$, also $s \neq s_0$ must correspond to the element $0 \in H^3(T^6, \mathbb{Z}_2)$, i.e.

$$u = u_0 \tag{A.76}$$

for all $s \in \mathcal{S}$. This independence of u on $s \in \mathcal{S}$ for the case of $M = T^6$ was first noted in [18], and was given a deeper explanation in [19].

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